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ON THE INTERACTION OF A TWO-DIMENSIONAL JET WITH A PARALLEL FLOW

by

Robert C. Ackerberg and Alexander Pal

September 1965

POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
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ERRATA

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Page 4, Eq. (2.1) should read: $\partial q_1/\partial n = q_1/R_1$.
Eq. (2.5) should read: $a(s) = \ln q_{\infty 1} + \ln [R(s) - d(s)]$.

Page 5, 13 lines down: "stramline" should be changed to "streamline".
Page 6, 3 lines down: Equation number (2.12) should be changed to (2.11).
Page 7, 8 lines down: Equation number (2.13) should be changed to (2.12).
Page 8, Eq. (3.12): Superscript hats (^) should be inserted over φ and ψ.
Page 12, 4 lines down should read: "late here, in a slightly....."
10 lines down should read: "bounded and $\nabla Q^*(\varphi, \psi) = o(1/r)$...."
Page 13, 2 lines up should read: ".... values, and Q is".
Page 14, line 6 should read:

$$D_\epsilon^\lambda [u, v] = \iint_{G_\epsilon^\lambda} \nabla u \nabla v \, d\varphi d\psi$$

Page 14, Eq. (5.2) should read:

$$4 \Delta_\epsilon^\lambda [u, v] = \iint_{G_\epsilon^\lambda} \nabla u \nabla v \, d\varphi d\psi$$

$$+ 1/(2\pi\epsilon^2 \ln \epsilon) \int_{C_\epsilon} u \, |dw| \int_{C_\epsilon} v \, |dw|$$

Page 15, Footnote, line 2 should read:

$$P \int_{-\infty}^{\infty} u(\varphi) \, d\varphi = \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right\} u(\varphi) \, d\varphi$$

Page 16, 3 lines up should read: "....VAR, using (5.6) as the....".
Page 17, 1 line down should read: ".... to the absence of any ....".
Page 21, Eq. (6.5')should read:

$$\tilde{\Delta} [u] = D [u_o] + 2(a/\pi) \int_0^{\pi/2} u_o(1, 2t) \, dt$$

Page 22, Eq. (6.8) should read:

$$\tilde{\Delta} [u] = \int_0^{\pi/2} u_o(1, 2t) [u_{o\rho}(1, 2t) + 2a/\pi] \, dt$$

Page 36, 2 lines down should read: ".... numerically using the"
Page 38, 2 lines up should read: ".... property (c) it follows...."

ON THE INTERACTION OF A TWO-DIMENSIONAL JET WITH A PARALLEL FLOW

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Robert C. Ackerberg and Alexander Pal

Contract No. DA 44-177-AMC-91(T)

Polytechnic Institute of Brooklyn

Department

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PIBAL Report No. 88

ON THE INTERACTION OF A TWO-DIMENSIONAL JET WITH A PARALLEL FLOW+

by

Robert C. Ackerberg* and Alexander Pal**
Polytechnic Institute of Brooklyn

SUMMARY

The injection of a two-dimensional jet of total head H_1 from an infinite plate into a uniform stream of lower total head H_2 is considered, the fluids being assumed inviscid and incompressible. Steady, irrotational solutions of Euler's equations are found for $H_1/H_2 \to \infty$. The region behind the jet is treated as a stagnant wake with constant pressure equal to that of the undisturbed stream, and the jet injection angle is fixed across the jet opening.

When a thin jet approximation is combined with Bernoulli's principle, a non-linear boundary condition is derived along the vortex sheet separating the jet from the external flow. The resulting non-linear potential problem (in the plane of the complex velocity potential) for the external flow is shown by Pal to be equivalent to a variational problem. A numerical procedure based on the variational principle and the Ritz-Galerkin method is used to solve for the case of normal injection with a digital computer. The pressure distribution along the plate upstream and streamline diagrams are given.

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* Assistant Professor, Dept. of Aerospace Engineering and Applied Mechanics.

** Research Associate, Dept. of Aerospace Engineering and Applied Mechanics.

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PART I

by

Robert C. Ackerberg and Alexander Pal

1. INTRODUCTION

This investigation is concerned with the injection of a jet of total head H_1 from an infinite plate into a uniform stream of lower total head H_2 (see Fig. 1a). Two-dimensional, steady, irrotational solutions of Euler's equations will be sought, the fluids being assumed inviscid and incompressible. The injection angle $-\alpha$ is fixed across the jet opening, and at large distances from the origin all motion is assumed to be directed in the positive x-direction. The region to leeward of the jet will be treated a a stagnant wake with constant pressure equal to that of the undisturbed stream.

Flows of this type occur frequently in connection with VTOL aircraft, ground effect machines and jet-flapped wings. The basic difficulty in their analytical study is the non-linear boundary condition which must be applied along a streamline separating regions of different total heads. To ensure the continuity of the static pressure a vortex sheet coincident with this streamline must be introduced. The extensive literature dealing with this subject assumes one or more of the following: 1) $H_1 = H_2$, 2) the jet is bounded by free streamlines and walls and there is essentially no interaction with the external flow, and 3) the jet injection angle is small and linearized flow is assumed.

Previous work using this model for 90° injection has been carried out by Taylor[1], Ting, Libby and Ruger[2], and Vizel and Mostinskii[3]. Taylor uses a rough theoretical analysis to determine the shape of the bounding streamline OC near the jet exit and points out the difficulty of verifying any results experimentally because the jet will fill a wedge of nearly 40° due to viscous spreading. Ting et al. considers the problem when H_1 is slightly larger than H_2 and uses ordinary perturbation techniques to correct the solution obtained for equal total heads. They noted the difficulty of a straightforward perturbation procedure due to a non-uniformity in the flow deflection angle along the streamline OC [near the jet exit] when $(H_1 - H_2) \to 0$. Vizel and Mostinskii treat the problem semi-empirically by introducing a coefficient of bending which is equivalent to specifying an average radius of curvature over the length of the jet. They compare their results with experimental data obtained in one of their references.†

The fundamental assumption of this analysis will be that the ratio of jet thickness d to its radius of curvature R will be small. Physically this is to be expected when $H_1/H_2 \to \infty$. Using the simplified flow of Fig. 1, it is then possible to take the basic non-linearity of the problem into account.‡

This report is presented in three parts. In Part I (by R. C. Ackerberg and A. Pal) a non-linear boundary condition is derived for the external flow based on the assumption that $d/R << 1$ inside the jet (§2). Since the jet shape is unknown, the analysis is similar to potential flows with free streamlines and is readily carried out in the plane of the complex velocity potential. In §3, a non-linear potential problem is derived in this plane for the logarithm of the speed along any streamline in the external flow. Formal asymptotic solutions of the potential problem have been found in §4 which are valid close to and far from the jet exit.

In Part II (by A. Pal), the equivalence of the boundary value problem with a

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† We have not been able to locate a copy of this reference.

‡ Motivated by this study Ackerberg[4] has started with the assumption $H_1/H_2 \to \infty$ instead of $d/R << 1$. The two assumptions are shown to be equivalent to first order for all streamlines not too close to the jet exit. See the footnote on p. 5.

variational principle is discussed in §5. In §6, a numerical method of minimizing the variational integral based on the Ritz-Galerkin method is given. The resulting system of non-linear equations is solved numerically for 90° injection in §7 using Newton's method.

In Part III (by R. C. Ackerberg and A. Pal), the numerical results are compared with experimental and theoretical results of other workers.

Most results obtained in this analysis would be altered significantly by viscous effects arising from the shear layers along OC and AB. In addition, this flow is notoriously unstable and the assumption of steady flow could probably not be maintained in an experiment. Nevertheless, it is expected that inviscid results will be valid close to the jet exit where a determination of the forces on the plate can be made.

2. THE THIN JET APPROXIMATION

Introduce the complex space coordinate $z = x + iy$ with origin at point O (see Fig. 1a). Since the locations of the streamlines OC and AB are not known in advance, it is convenient to formulate this problem using a coordinate system based on the streamlines and equipotential lines. Define the complex velocity potential $w = \varphi + i\psi$, with $\psi = 0$, and $\psi = m > 0$ along OC and AB respectively, and $\psi < 0$ in the external flow. The vortex sheet along OC requires that two velocity potentials be used, and conditions in the jet and the external flow will be denoted by the subscripts 1 and 2, respectively.† The flow region in the z-plane maps into the w-plane as shown in Fig. 2.

Choose a curvilinear coordinate system (s, n) in the jet as follows (see Fig. 1b):

† Since ψ is continuous across the streamline OC, it need not be subscripted.

The n-coordinate of a point P in the jet will be the distance PQ measured along the φ = const. line through P, and the s-coordinate will be the arc-length OQ along the bounding streamline OC. The condition of irrotationality requires

$$\frac{\partial q_1}{\partial n} = \frac{q_1}{R_1} , \tag{2.1}$$

where R_1 is the radius of curvature of the streamline through (s, n). When $d/R_1 << 1$, we may with good approximation assume

$$R_1(s, n) \approx R(s) - n , \tag{2.2}$$

where R(s) is the radius of curvature of the streamline OC. Physically this means that the streamlines in the jet crossing an equipotential line are approximated by concentric circles. Substituting (2.2) into (2.1) and integrating yields

$$\ln q_1(s,n) = - \ln[R(s) - n] + a(s) , \tag{2.3}$$

where a(s) is an arbitrary function. Along the free streamline AB we require

$$q_1(s, n) = q_{\infty 1} \quad \text{for } n = d(s) . \tag{2.4}$$

Substituting (2.4) in (2.3) we obtain

$$a(s) = \ln q_{\infty 1} + \ln[R(s) - d(s)] \tag{2.5}$$

Therefore,

$$q_1(s, n)/q_{\infty 1} = \frac{R(s) - d(s)}{R(s) - n} ; \tag{2.6}$$

thus, at any fixed value of s the velocity distribution is that of a potential vortex.[†]

The conservation of mass in the jet requires that a fixed volumetric flow rate cross every section s = constant, i.e.,

$$\int_C^{d(s)} q_1(s,n)\,dn = q_{\infty 1} d_\infty, \tag{2.7}$$

where $d_\infty = d(\infty)$ is the uniform jet thickness far downstream. Substituting (2.6) in (2.7) and integrating yields

$$-\left[R(s) - d(s)\right] \ln \frac{R(s) - d(s)}{R(s)} = d(s)\left[1 - \frac{d(s)}{R(s)} + \dots\right] = d_\infty, \tag{2.8}$$

i.e.,

$$d(s) = d_\infty\left[1 + O\left(\frac{d}{R}\right)\right] \approx d_\infty = d. \tag{2.9}$$

Thus, the thickness of the jet is constant to first order.

Across the streamline OC the pressure must be continuous. Using Bernoulli's principle this requires

$$\rho_1 \hat{q}_1^{\,2} - \rho_2 \hat{q}_2^{\,2} = 2(H_1 - H_2), \tag{2.10}$$

where the ^ denotes the velocities at some point along the bounding stramline OC [e.g., $\hat{q}_1 = q_1(s,o)$]. Here $H_{1,2}$ are the total heads in the jet and the external flow, respectively. When (2.10) is applied at an infinite distance downstream, a relationship

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† When $\alpha \neq \pi/2$, Ackerberg[4] has shown that this approximation is in error by $O(d/R)$ near the jet exit $[|z| = O(d)]$ as a result of neglecting a boundary condition along OA. The thin jet approximation is valid in the region $|z| = O(d/\mu)$, where $\mu = \rho_2 q_{\infty 2}^2 / \rho_1 q_{\infty 1}^2 \to 0$. Nevertheless, the motion in the external flow (i.e., the solution of Eqs. (3.13)-(3.17)) will be determined correctly to o(1) up to the jet exit because the inaccuracy of (2.6) modifies the boundary condition along OC only to a small order.

between the basic parameters is obtained. Introducing the small dimensionless parameter $\mu = (\rho_2 q_{\infty 2}^2 / \rho_1 q_{\infty 1}^2)$ we obtain

$$\frac{H_1 - H_2}{\frac{1}{2}\rho_1 q_{\infty 1}^2} = 1 - \mu \ . \tag{2.12}$$

Substituting (2.6) with n = 0 into (2.10) and neglecting terms of $O[(d/R)^2]$ yields

$$\left(\frac{\hat{q}_2}{q_{\infty 2}}\right)^2 = 1 - \frac{2d}{\mu R(s)} \ , \tag{2.12}$$

This is a boundary condition along OC for the external flow which relates the speed and curvature along this streamline.

3. THE FLOW IN THE EXTERNAL STREAM†

In the external flow introduce the complex velocity potential

$$w(z) = \varphi + i\psi \ , \tag{3.1}$$

where $\psi \leq 0$, $-\infty < \varphi < \infty$. The complex velocity is

$$\frac{dw}{dz} = u - iv = q\, e^{-i\theta} \ . \tag{3.2}$$

The boundary value problem for the external flow is most readily formulated in the w - plane. Introduce the logarithm of the complex velocity

$$\Gamma(w) = \ln\left(\frac{1}{q_{\infty 2}} \frac{dw}{dz}\right) = Q(\varphi, \psi) - i\theta(\varphi, \psi) \ . \tag{3.3}$$

Here Q and θ are conjugate harmonic functions related by the Cauchy-Riemann

† Hereafter the subscript 2 which identifies quantities in the external flow will be omitted whenever possible.

equations

$$\frac{\partial Q}{\partial \varphi} = - \frac{\partial \theta}{\partial \psi} , \tag{3.4a}$$

$$\frac{\partial Q}{\partial \psi} = \frac{\partial \theta}{\partial \varphi} , \tag{3.4b}$$

and defined in the region $\psi \leq 0$, $-\infty < \varphi < \infty$.

The curvature of any streamline is given by

$$\frac{1}{R(\varphi, \psi)} = \frac{\partial \theta}{\partial s} = q \frac{\partial \theta}{\partial \varphi} = q_{\infty 2} e^{Q} \frac{\partial Q}{\partial \psi} , \tag{3.5}$$

where we have used (3.4b) and the relation $q = q_{\infty 2} e^{Q}$. Applying (3.5) to the bounding streamline OC and substituting in (2.13) yields [note $\hat{q}_2 = q_{\infty 2} e^{Q(\varphi, 0)}$]

$$e^{2Q(\varphi, 0)} = 1 - \frac{2 d q_{\infty 2}}{\mu} e^{Q(\varphi, 0)} \left(\frac{\partial Q}{\partial \psi}\right)_{\psi = 0} \tag{3.6}$$

Solving for $\partial Q / \partial \psi$ we obtain

$$\left(\frac{\partial Q}{\partial \psi}\right)_{\psi = 0} = \frac{\mu}{2 d q_{\infty 2}} \left[e^{-Q(\varphi, 0)} - e^{Q(\varphi, 0)} \right]$$

$$= - \frac{1}{m} \frac{\rho_2 q_{\infty 2}}{\rho_1 q_{\infty 1}} \sinh Q(\varphi, 0) . \tag{3.7}$$

Along the wall DO the deflection is fixed. Thus,

$$\theta(\varphi, 0) = 0 \quad \text{along } \psi = 0, \ \varphi < 0 , \tag{3.8}$$

This may be written in terms of Q using (3.4b), i.e.,

$$\frac{\partial Q}{\partial \psi} = 0 \quad \text{along } \psi = 0, \ \varphi < 0 . \tag{3.9}$$

At infinity the flow must be uniform and undeflected. Therefore,

$$\theta(\varphi, \psi) \to 0, \quad Q(\varphi, \psi) \to 0, \quad \text{as} \quad |w| \to \infty . \tag{3.10}$$

Eqs. (3.7)-(3.10) are satisfied by the function $\Gamma(w) \equiv 0$, corresponding to a uniform stream and for which the flow does not turn through an angle $-\alpha$ near point O. To satisfy this condition Γ must have a logarithmic singularity at $z = 0$ (i.e., $w = 0$) corresponding to a stagnation point flow in a corner. Thus

$$\Gamma(w) = \frac{\alpha}{\pi} \ln (e^{\pi i} w) + \Gamma^*(w) , \tag{3.11}$$

where $0 \geq \arg w \geq -\pi$, and $\Gamma^*(w)$ is bounded at $w = 0$.

On introducing the non-dimensional, scaled complex velocity potential

$$\hat{w} = \varphi + i \psi = \frac{\rho_2 q_{\infty 2}}{\rho_1 q_{\infty 1}} \frac{w}{m} , \tag{3.12}$$

in Eqs. (3.7)-(3.11), the following non-linear potential problem is obtained for Q_2:

$$\nabla^2 Q = 0 \quad \text{for} \quad \hat{\psi} < 0, \ -\infty < \hat{\varphi} < \infty , \tag{3.13}$$

$$\frac{\partial Q}{\partial \hat{\psi}} = 0 \quad \text{for} \quad \hat{\psi} = 0 , \ \hat{\varphi} < 0 , \tag{3.13}$$

$$\frac{\partial Q}{\partial \hat{\psi}} = - \sinh Q \quad \text{for} \quad \hat{\psi} = 0, \ \hat{\varphi} > 0, \tag{3.15}$$

with

$$Q(\hat{\varphi}, \hat{\psi}) - \frac{\alpha}{\pi} \ln |\hat{w}| \tag{3.16}$$

bounded at $\hat{\varphi} = \hat{\psi} = 0$, and

$$Q(\hat{\varphi}, \hat{\psi}) \to 0 \quad \text{for} \quad |\hat{w}| \to \infty . \tag{3.17}$$

4. ASYMPTOTIC EXPANSIONS OF $\hat{\Gamma}(\hat{w})$

Formal asymptotic expansions of $\hat{\Gamma}(\hat{w})$ which are analytic and which satisfy the boundary conditions term by term without inconsistency, were constructed for $\hat{w} \to 0$, and $\hat{w} \to \infty$. Denoting $e^{\pi i} \hat{w} = W$, we obtain for $W \to 0$,

$$\hat{\Gamma}(-W) \sim \frac{\alpha}{\pi} \ln W + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_{mn}(\ln W) W^{n+m(\alpha/\pi)}, \tag{4.1}$$

where d_{mn} is in general a polynomial in $\ln W$ with real coefficients. Coefficients have been obtained recursively up to n = 2 with no inconsistencies provided a single logarithmic term $W^2 \ln W$ is introduced. The coefficients which multiply integral powers of W are indeterminate by this formal procedure.

If $\alpha = \pi/2$, the following simpler expression is obtained:

$$\hat{\Gamma}(-W) \sim \frac{1}{2} \ln W + \sum_{n=0}^{\infty} a_n W^{n/2}, \tag{4.2}$$

where a_0 and a_2 are undetermined and $a_1 = -e^{-a_0}$. From the numerical computations in Part II for $\alpha = \pi/2$, $a_0 = 0.0450\ldots$, $a_1 = -0.955\ldots$, and $a_2 = -1.58\ldots$ Note that the relationship $a_1 = -e^{-a_0}$ is satisfied with good approximation.

In the case $W \to \infty$, the form of the formal asymptotic expansion does not depend on α. We find

$$\hat{\Gamma}(-W) \sim \sum_{n=0}^{\infty} P_n(\ln W) W^{-(n+1/2)}, \tag{4.3}$$

where $P_n(L)$ denotes a polynomial (which seems to be of order n) with real coefficients. From the first few terms it appears that the P_n's can be obtained recursively with the exception of their constant terms. Thus,

$$\hat{\Gamma}(-W) \sim A_0 W^{-1/2} + \left(-\frac{1}{2\pi} A_0 \ln W + A_1\right) W^{-3/2} + \ldots \tag{4.4}$$

Numerical computations for $\alpha = \pi/2$ yield $A_0 = -0.797\ldots$ *

The equations of the streamlines may be found by integrating (3.3), i.e.,

$$\frac{\mu}{d} z = \int_0^{\hat{w}} e^{-\hat{\Gamma}(\hat{w})} d\hat{w} . \tag{4.5}$$

Putting $\hat{w} = \hat{\varphi} > 0$ in (4.2) and (4.4) and substituting these results in (4.5), we obtain the asymptotic behavior of the streamline OC close to and far from the jet exit. These results are readily expressed in terms of the nondimensional, scaled complex space coordinate

$$Z = X + iY = \frac{\mu}{d} z , \tag{4.6}$$

where z denotes the complex coordinate of a point on OC. For $|z| \to 0$ in the special case $\alpha = \pi/2$,

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* Although these expansions are derived in a purely formal manner, theoretical results of Pal[5] suggest that they are indeed asymptotic to $\Gamma(w)$. It is proved there that Q is asymptotic to the leading terms of the expansions, and formulas are obtained expressing d_{00} and A_0 with integrals involving the boundary values of Q. Note that $Q(\varphi, 0) = O(\varphi^{-3/2})$ for $\varphi \to +\infty$ whereas $\Gamma(w) = O(|w|^{-1/2})$ for $|w| \to \infty$ in agreement with the formal asymptotic expression (4.4).

$$X \sim \frac{1}{4} Y^2 + O(Y^4), \tag{4.7}$$

and for $|z| \to \infty$,

$$X \sim \left(\frac{Y + \frac{\pi}{2} A_0^2}{2 A_0} \right)^2 - \frac{1}{2} A_0^2 \ln \left(\frac{Y + \frac{\pi}{2} A_0^2}{2 A_0} \right)^2 + \text{const.} + o(1) . \tag{4.8}$$

Using only the first term of (4.7), and the first term of (4.8) with an experimentally determined constant added, these asymptotic formulae are plotted along with the numerical result for $\alpha = \pi/2$ in Fig. 3. The asymptotic curves fare in well with the numerical result.

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PART II

by

Alexander Pal*

*The author is indebted to Dr. R.C. Ackerberg for his cooperation in the research project out of which this section grew. Frequent discussions with him of details of this work and his careful review of this section are considered especially helpful. He also helped by programming some of the subroutines used in the numerical work.

5. VARIATIONAL PRINCIPLE

In Part I of this paper it was shown that the fluid flow problem of the interaction of a fast jet with a potential flow, which is a parallel flow at large distance from the orifice can be reduced to the solution of a plane boundary value problem which we formulate there, in a slightly different way, using reflection on the φ-axis.

Let G denote the whole (φ, ψ) plane, slit along the $-\varphi$-axis. Then a function $Q(\varphi, \psi)$ has to be found which satisfies the following conditions:

Conditions in G

(a) $Q(\varphi, \psi)$ is harmonic in G and symmetric to the real axis.

(b) The function $Q^*(\varphi, \psi) = Q(\varphi, \psi) - (\alpha/\pi) \ln r \; (r = (\varphi^2 + \psi^2)^{\frac{1}{2}})$ is bounded and $\nabla Q^*(\varphi, \psi) = o(r)$ uniformly as $r \to 0$.

(c) $Q(\varphi, \psi) \to 0$, $\nabla Q(\varphi, \psi) = o(1/r)$ uniformly, as $r \to \infty$.

Boundary Conditions

(d) Q and $\frac{\partial Q}{\partial n}$ are continuous on the $+\varphi$-axis. Here n is the outer normal to the domain G on $\psi = 0$, i.e., $\frac{\partial Q}{\partial n} = \frac{\partial Q}{\partial \psi}$ on the lower side and $= -\frac{\partial Q}{\partial \psi}$ on the upper side of the cut.

(e) $\frac{\partial Q}{\partial n} = -\sinh Q$ on $\psi = 0$, $\varphi > 0$. (5.1)

(f) $Q(\varphi, 0) = O(\varphi^{-3/2})$ as $\varphi \to +\infty$.

Conditions (b), (c), and (f) are more stringent than necessary. It was shown[5] that the problem has a unique solution under much milder conditions (see also footnote on page 16). However, since this solution satisfies the stricter conditions above (including (b), (c), and (f)), we may use this formulation for convenience. It should also be noted that the formal asymptotic expansions obtained in Part I are consistent with these conditions.

Functions which satisfy conditions (b), (c), (d), and (f), (but are not necessarily harmonic or satisfy the boundary conditions) will be called "admissible", and the family of admissible functions with a fixed value of α will be denoted by $\mathcal{A}_\alpha$.

An important property of the family $\mathcal{A}_\alpha$ is: If the functions u, v belong to $\mathcal{A}_\alpha$, then for all $\lambda (0 \leq \lambda \leq 1)$,

$$w = \lambda u + (1 - \lambda) v$$

also belongs to $\mathcal{A}_\alpha$. This may be expressed concisely as follows: <u>The function space $\mathcal{A}_\alpha$ is convex.</u>* The statement is obvious; to see it one only needs to verify that w satisfies all criteria of admissibility if u and v do.

The main difficulty in solving the above boundary value problem stems, of course, from the non-linearity of the boundary condition (5.1). The fact that Q is harmonic suggests a variational method based on the Dirichlet principle.[6**]

<u>Generalized Dirichlet Integral.</u> We first extend the notion of the Dirichlet integral slightly to admit functions which have a logarithmic singularity at the origin, such that

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*The name convex is suggested by the analogy to finite dimensional spaces; e.g., a ball is convex because, if the points p, q belong to the ball, then the entire straight segment connecting p and q belongs to the same ball.

**The Dirichlet principle states that in a domain G with a sufficiently smooth boundary, there is a function Q which makes the Dirichlet integral

$$D[Q] = \frac{1}{4} \iint_G (\nabla Q)^2 \, d\varphi \, d\psi$$

minimum under the constraint that Q assumes given continuous boundary values, Q is harmonic in G.

they satisfy condition (b).

We will use the following notations:

1. G_ϵ^λ will denote the subdomain $\epsilon < r < \lambda$ of G;

2. $D_\epsilon^\lambda[u]$, $D_\epsilon^\lambda[u, v]$ will denote the Dirichlet integral of u over the domain G_ϵ^λ, and the associated bilinear form, i.e.

$$D_\epsilon^\lambda[u, v] = \iint_{G_\epsilon^\lambda} \nabla u \, \nabla v \, d\varphi \, d\psi ,$$

$$D_\epsilon^\lambda[u] = D_\epsilon^\lambda[u, u] .$$

3. 0 subscripts and ∞ superscripts will be omitted in these forms, as from Δ_ϵ^λ defined below.

4. C_λ will denote the circle $r = \lambda$.

The Dirichlet integral is for admissible functions in general divergent. In order to separate its "finite part", we define

$$\Delta_\epsilon^\lambda[u, v] = \iint_{G_\epsilon^\lambda} \nabla u \, \nabla v \, d\varphi \, d\psi + \frac{1}{2\pi \epsilon^2 \ln \epsilon} \int_{C_\epsilon} u \, |dw| \int_{C_\epsilon} v \, |dw| , \tag{5.2}$$

and

$$\Delta_\epsilon^\lambda[u] \equiv \Delta_\epsilon^\lambda[u, u] .$$

It is easy to see that if $u(\varphi, \psi)$, $v(\varphi, \psi)$ satisfy conditions (b) and (c), then $\Delta_\epsilon^\lambda[u, v]$ has a limit $\Delta[u, v]$ when $\epsilon \to 0$ and $\lambda \to \infty$, (and similarly $\Delta_\epsilon^\lambda[u] \to \Delta[u] \equiv \Delta[u, u]$).

If all integrations in the preceeding definitions are extended only to the upper (lower) halfplane, we can define the functionals $\Delta_{\epsilon,+}^\lambda[u]$, $\Delta_+[u]$, etc. ($\Delta_{\epsilon,-}^\lambda[u]$, $\Delta_-[u]$ etc.) completely analogously. Thus

$$\Delta[u] = \Delta_+[u] + \Delta_-[u] \quad \text{etc.}$$

We mention some important properties of the functional $\Delta[u]$:

(1) Let the functions $u(\varphi, \psi)$, $v(\varphi, \psi)$ be admissible, say, $u \in \mathcal{a}_\alpha$, $v \in \mathcal{a}_\beta$. Then $u+v \in \mathcal{a}_{\alpha+\beta}$ and

$$\Delta[u+v] = \Delta[u] + 2\Delta[u, v] + \Delta[v] \tag{5.3}$$

Eq. (5.3) is an immediate consequence of the identity

$$\Delta_\epsilon[u+v] = \Delta_\epsilon[u] + 2\Delta_\epsilon[u, v] + \Delta_\epsilon[v],$$

if we observe that $\Delta_\epsilon[u], \Delta_\epsilon[v], \Delta_\epsilon[u+v]$ all have limits as $\epsilon \to 0$.

(2) Suppose $h(\varphi, \psi)$ is a harmonic function in $\mathcal{a}_\alpha$, and $k(\varphi, \psi)$ is a function in $\mathcal{a}_0$, (thus k has a finite Dirichlet integral in G). Then,

$$4\Delta_-[h, k] = P\int_{-\infty}^{\infty} h_\psi(\varphi, 0)\, k(\varphi, 0)\, d\varphi\,.^* \tag{5.4}$$

The proof of Eq. (5.4) can be found in Appendix 1.

Substituting (5.4) into (5.3), we obtain

$$\Delta[h+k] - \Delta[h] = \frac{1}{2} P\int_{-\infty}^{\infty} h_\psi(\varphi, 0)\, k(\varphi, 0)\, d\varphi + D[k]\,. \tag{5.5}$$

(3) The functional $\Delta[u]$ can be defined alternatively by ordinary Dirichlet integrals. Let λ be any positive number. Then

$$\Delta[u] = D_\lambda[u] + D^\lambda[u^*] + \frac{\alpha}{2\pi\lambda}\int_{C_\lambda} u\, ds - \frac{\alpha^2}{2\pi}\ln\lambda \tag{5.6}$$

where $u^* = u - (\alpha/\pi)\ln r$. The proof of this identity can be found in Appendix 2.

Formulation of Variational Problem. If u is admissible, then the boundary integral

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* P denotes here the "Cauchy principal value" of the integral:

$$P\int_{-\infty}^{\infty} u(\varphi)\, d\varphi = \lim_{\epsilon \to 0}\left\{\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right\} u(\varphi)\, d\varphi\,.$$

$$B[u] = \int_0^{\infty} [\cosh u(\varphi, 0) - 1] \, d\varphi \tag{5.7}$$

is always convergent. From $\Delta[u]$ and $B[u]$ we form the composite expression

$$F[u] = \Delta[u] + B[u] . \tag{5.8}$$

We mention here a property of $F[u]$ which will be useful later: $F[u]$ is a <u>"convex" functional;</u> i.e., if $u \in \mathcal{A}_\alpha$, $v \in \mathcal{A}_\alpha$, and for some λ, $0 < \lambda < 1$, $w = \lambda u + (1 - \lambda) v$, $u \not\equiv v$, then

$$F[w] < \lambda F[u] + (1 - \lambda) F[v] .^{*} \tag{5.9}$$

The proof of this property can be found in Appendix 3.

We consider now the following variational problem (problem "VAR"): Find an 'admissible' function $Q(\varphi, \psi) \in \mathcal{A}_\alpha$, such that $F[Q] < F[Q_1]$ for any $Q_1 \in \mathcal{A}_\alpha$ different from Q. Although the existence of a function Q will not be established in this paper,** we will show that the problem VAR is <u>equivalent</u> to the problem BV; thus if problem BV has a solution, it is also a solution of problem VAR, and conversely. Such a solution is by its minimum property necessarily unique.***

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*Equation (5.9) is obviously satisfied by an (ordinary) function $F(u)$ which is convex from below. Moreover, convex functions can be defined by (5.9).

**The existence of a solution of the problem VAR is shown (see Pal[5]) in a stronger form. It is proven there that there is a unique function Q admissible in the above sense, which renders $F[Q]$ the smallest in a much wider function class $\tilde{\mathcal{A}}_\alpha$. ($\tilde{\mathcal{A}}_\alpha$ requires for membership $D^1[u^*] < \infty$, $D_1[u] < \infty$ and $B[u] < \infty$, instead properties (b), (c), (d), and (f).

A different proof of the equivalence of BV and VAR, using (1.6) as the defining equation of $\Delta[u]$ was given by Ackerberg.[4]

***I am thankful for this observation on the uniqueness to R. C. Ackerberg.

We call attention in this variational problem to the absense of any kind of boundary conditions along the real axis. The fulfillment of the boundary condition will be shown to be a consequence of the minimum property of the solution. Similar situations are not infrequent in variational problems. ("Free boundary values", "natural boundary conditions")[6].

Proof of the equivalence of the problems BV and VAR

I. We first show that if the function $Q(\varphi, \psi)$ satisfies the conditions of VAR, then it is also a solution of BV.

(a) The function Q must be harmonic in G. If it is not, then there is a circular disk ω entirely in G, in which Q is not harmonic. Then it is possible by Poisson's integral to construct a function Q_1 such that $Q = Q_1$ on the boundary and outside ω, and Q_1 harmonic in ω. Then by Dirichlet's principle,

$$D_\omega[Q_1] < D_\omega[Q] .$$

(D_ω denotes here the Dirichlet - integral in ω). If ε is chosen so small that C_ε is completely outside ω,

$$\Delta_\varepsilon[Q_1] < \Delta_\varepsilon[Q]$$

since $Q = Q_1$ outside ω. Hence

$$\Delta[Q_1] < \Delta[Q] ,$$

a contradiction to the assumption that Q satisfies the conditions of problem VAR.

(b) The function Q must be symmetric to the φ-axis. Suppose e.g. that

$$\Delta_+[Q] \geq \Delta_-[Q] .$$

Then we define the new admissible function

$$Q_1(\varphi, \psi) = \begin{cases} Q(\varphi, -\psi) & \text{if } \psi > 0 , \\ Q(\varphi, \psi) & \text{if } \psi \leq 0 . \end{cases}$$

Clearly

$$\Delta[Q_1] \leq \Delta[Q]$$

which by assumption is only possible if $Q_1 = Q$.

(c) We will use the definition of the first variation of functionals customary in variational calculus. If $\delta Q(\varphi, \psi)$ is an arbitrary* variation of Q, and $\Phi[Q]$ is a functional, then

$$\delta \Phi[Q] = \left\{ \frac{\partial}{\partial \lambda} \Phi[Q + \lambda \delta Q] \right\}_{\lambda = 0} \tag{5.10}$$

is the "first variation" of $\Phi[Q]$. Thus we show that

$$\delta F[Q] = P \int_{-\infty}^{+\infty} [Q_\varphi(\varphi, 0) + \eta(\varphi) \sinh Q(\varphi, 0)] \, \delta Q(\varphi, 0) \, d\varphi \tag{5.11}$$

where $\eta(\varphi) = 1$, if $\varphi > 0$ and $\eta(\varphi) = 0$ if $\varphi < 0$.

Equation (5.11) immediately implies that if Q is a solution of the problem VAR, then

$$Q_\varphi(\varphi, 0) = \begin{cases} 0 & \text{if } \varphi < 0, \\ -\sinh Q(\varphi, 0) & \text{if } \varphi > 0, \end{cases}$$

i.e., Q satisfies all conditions of BV.

Equation (5.11) is an easy consequence of (5.5). Substituting $h = Q$, $k = \lambda \delta Q$ in (5.5) and taking into account the symmetry of $Q, \delta Q$,

$$\Delta[Q + \lambda \delta Q] - \Delta[Q] = \lambda \int_0^\infty Q_\varphi(\varphi, 0) \delta Q(\varphi, 0) + \lambda^2 D[\delta Q].$$

This implies by definition (1.10) that

$$\delta \Delta[Q] = \int_0^\infty Q_\varphi(\varphi, 0) \, \delta Q(\varphi, 0) \, d\varphi. \tag{5.12}$$

On the other hand, from the definition of $B[Q]$ (Eq. (5.7)),

$$\delta B[Q] = \int_0^\infty \sinh Q(\varphi, 0) \, \delta Q(\varphi, 0) \, d\varphi$$

---

*However, in the present context, only such variations δQ are admissible for which $Q + \lambda \delta Q \in \mathcal{Q}_\alpha$ for all sufficiently small values of $|\lambda|$. For this it is sufficient if $\delta Q \in \mathcal{Q}_0$. Furthermore if is sufficient to consider only variations symmetric to the φ-axis.

is obvious. Combining with (5.12), we obtain (5.11).

II. Suppose now that $Q(\varphi, \psi)$ is a solution of problem BV. Then by (5.11), $\delta F[Q] = 0$, i.e., $F[Q]$ is stationary. To show that $F[Q]$ has an absolute minimum in Q, we will use the "convexity" of the function space $\mathcal{A}_\alpha$ and of the functional F.

Let $Q_1 \not\equiv Q$ be any other admissible function. Consider the functions

$$S(\varphi, \psi; \lambda) = (1-\lambda)Q + \lambda Q_1 . \quad (0 \leq \lambda \leq 1) .$$

By the convexity of $\mathcal{A}_\alpha$, S also belongs to $\mathcal{A}_\alpha$. Denote $F[S] = \Phi(\lambda)$. $\delta F[Q] = 0$ implies $\Phi'(0) = 0$. However, by the convexity of $F[u]$, the function $\Phi(\lambda)$ is convex from below in (0, 1); thus,

$$F[Q_1] \equiv \Phi(1) > \Phi(0) \equiv F[Q] ,$$

i.e., $F[u]$ has an absolute minimum when $u = Q$.

6. CONFORMAL MAPPING INTO A CIRCLE

In the numerical solution of the variational problem VAR it is desirable that the domains of integration be finite. Therefore, the half-plane $\psi < 0$ is conformally mapped into the half-circle disk $\tilde{G}\left\{s : |s| < 1,\ \mathrm{Im}\ s > 0\right\}$ of the complex s-plane. The mapping function is

$$s = S(w) = \frac{(-w)^{\frac{1}{2}} + 1}{(-w)^{\frac{1}{2}} - 1} ; \tag{6.1a}$$

and its inverse

$$w = W(s) = -\left(\frac{s+1}{s-1}\right)^2 . \tag{6.1b}$$

Here $(-w)^{\frac{1}{2}}$ is defined by providing the w-plane with a slot along the positive imaginary axis; thus, $0 \leq \arg(-w)^{\frac{1}{2}} \leq \frac{\pi}{2}$.

If w	Then S(w)
$=0$	$= -1$
on the $-\varphi$- axis	on the $(-1, +1)$ diameter
$= \infty$	$= +1$
on the $+\varphi$-axis	on the $s = e^{i\tau}$, $0 < \tau < \pi$ half circle
$+1$	i

(see Fig. 4)

Let $s = \rho e^{i\tau}$. The mapping of the positive φ-axis into the half-circle $\rho = 1$, $0 < \tau < \pi$ is described by

$$\varphi = [\cot(\tau/2)]^2 . \tag{6.2}$$

Evaluation of $F[u]$ in the s-plane. To take the logarithmic singularity of the function $\Gamma(w) = Q - i\rho$ at the origin of the w-plane into account, we introduce the functions

$$R(s) = \frac{2\alpha}{\pi} \ln \frac{s+1}{2}, \tag{6.3}$$

$$r(\rho,\tau) = \operatorname{Re} R(s) = \frac{\alpha}{\pi} \ln \frac{\rho^2 + 2\rho\cos\tau + 1}{4}. \tag{6.3'}$$

Write $\tilde{\Gamma}(s) = \Gamma(W(s))$ and

$$\Gamma_0(s) = \tilde{\Gamma}(s) - R(s), \tag{6.4}$$

$$Q_0(\rho,\tau) = \tilde{Q}(\rho,\tau) - r(\rho,\tau). \tag{6.4'}$$

In general, for any admissible $u(\rho,\tau)$, the function $u_0(\rho,\tau) = u - r(\rho,\tau)$ is bounded in $\rho \leq 1$. Further conditions on u_0 are specified below.

The functional $\Delta[u]$ can be transformed into the s-plane using (5.5). Replacing there h by $r(\rho,\tau)$, k by u_0 and observing that $r_\rho(1,\tau) = \alpha/\pi$, we obtain

$$\Delta[u] = D[u_0] + \frac{\alpha}{\pi}\int_0^{\pi} u_0(1,\tau)\,d\tau + \Delta[r], \tag{6.5}$$

Since the Dirichlet integral is invariant with respect to conformal mapping, the term $D[U_0]$ may be evaluated in the s-plane. $\Delta[r]$, a constant, is irrelevant with respect to the variational problem. Therefore we will use the form

$$\tilde{\Delta}[u] = D[u_0] + \int_0^{\pi/2} u_0(1,2t)\,dt, \tag{6.5'}$$

$$\tilde{F}[u] = \tilde{\Delta}[u] + \tilde{B}[u], \tag{6.6}$$

where

$$\tilde{B}[u] = 2\int_0^{\pi/2} [\cosh u(1,2t) - 1]\frac{\cos t}{\sin^3 t}\,dt$$

$$= \int_0^{\pi/2} \left\{ \exp[\tfrac{1}{2}u_0(1,2t)]\cos t - \exp[-\tfrac{1}{2}u_0(1,2t)]\right\}^2 \frac{dt}{\sin^3 t}. \tag{6.7}$$

When u_o is harmonic, (6.5′) can be expressed in the form of boundary integrals by transforming $D[u_o]$ with Green's formula:

$$\tilde{\Delta}[u] = \int_0^{\pi/2} u_o(1,2t)[\, u_{o\varphi}(1,2t) + 1\,]\,dt \quad . \tag{6.8}$$

<u>Boundary conditions at $s=+1$ and $s=-1$.</u> $Q(\varphi, \psi)$ is expected to possess properties (b), (c) and (f) of "admissible" functions. In the s-plane this can be expressed as follows.

(b′) $Q_o(1, \pi) = 0$, and $\nabla Q_o = o(|s+1|^{-1})$

as $s \to -1$. (Here as above, $Q_o(\rho, \tau) = Q(\rho, \tau) - r(\rho, \tau)$.)

(c′) $Q_o(1, 0) = 0$, and $\nabla Q_o = o(|s-1|^{-1})$ as $s \to +1$.

(f′) $Q(1, \tau) = O(\tau^3)$ (6.9)

as $\tau \to 0$. This implies

$$Q_o(1, \tau) = \frac{\alpha}{4\pi}\tau^2 + O(\tau^3) \tag{6.10}$$

as $\tau \to 0$.

7. APPLICATION OF THE RITZ-GALERKIN METHOD

The variational problem VAR was solved in the special case $\alpha = \pi/2$ (normal injection) numerically, with the Ritz-Galerkin method. This method is often used with advantage when the value of the variational integral is the most important result, e.g., in problems involving eigenvalues (elastic buckling, resonant frequencies in vibration problems). It is known from experience that usually the value of the variational integral can be determined much more precisely than the minimizing function. In the present case only the minimizing function is of physical significance. Nevertheless, the Ritz-Galerkin method (referred to by R - G) was found quite suitable and probably better than a relaxation type method. Our choice fell on the former for the following reasons:

(1) In the R - G method the unknown function is approximated with a linear combination of known functions (we will say that these functions form a "base"). The base can be chosen in such a manner that they contain much of the information known about the solution before the computation is started. In the present problem it is known that the solution is harmonic, symmetric to the real axis, and smooth on the boundary. The base can be chosen to satisfy these conditions identically. Boundary conditions at $s = -1$ and $s = +1$ can be taken into account by proper combination of the elements of the base. Thus a significant portion of the computing work can be saved if known properties of the solution are taken into account by a judicious choice of the base.

In the present problem the R - G method requires the determination of a one-dimensional array of coefficients, (p_n in Eq. (7.2)) as opposed to the two-dimensional array of grid points in the relaxation methods. (Actual computation, discussed below, showed that only 15 coefficients yielded as satisfactory results as probably thousands of values in grid points would have done.)

(2) The circular shape of the domain makes the use of rectangular grids clumsy. On the other hand, if the w-plane is chosen, the infiniteness of the domain is a source of trouble.

According to the R-G method, we use a formal expansion

$$\Gamma_0(s) = \tilde{\Gamma}(s) - \ln \frac{s+1}{2} - \sum_{n=0}^{\infty} p_n H_n(s) , \tag{7.1}$$

$$Q_0(s) = \tilde{Q}(\rho, \tau) - \frac{1}{2} \ln \frac{\rho^2 + 2\rho \cos\tau + 1}{4}$$
$$= \sum_{n=0}^{\infty} p_n h_n(\rho, \tau) , \tag{7.2}$$

where $H_n(s)$ are analytic functions to be specified, $h_n(\rho, \tau) = \operatorname{Re} H_n(s)$, and the p_n are unknown coefficients. To take full advantage of the economy of the R-G method, the functions $h_n(\rho, \tau)$ will be chosen such that they satisfy the following conditions:

1) $h_n(\rho, \tau)$ is harmonic in $\rho < 1$, continuous on $\rho = 1$.

2) $h_n(\rho, \tau)$ is symmetric to the real s-axis.

3) $h_n(1, \tau)$ must form a complete system in $(0, \pi)$, i.e., every square integrable function $u(\tau)$ can be expanded in a series

$$u(\tau) = \sum_n \xi_n h_n(1, \tau) \quad .^{*}$$

$h_n(1, \tau)$ need not be differentiable in $\tau = 0, \pi$, since $Q_0(1, \tau)$ is not differentiable there.

The functions $h_n(\rho, \tau)$ will be determined such that on the half-circle $\rho = 1$, $0 \leq \tau \leq \pi$

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*This expansion need be valid only with the exception of a set of measure zero.

$$h_{2n}(1,\tau) = \cos n\tau \qquad (n = 0, 1, 2, \ldots), \tag{7.3'}$$

$$h_{2n-1}(1,\tau) = \sin n\tau \qquad (n = 1, 2, \ldots). \tag{7.3''}$$

$Q_0(1, \tau)$ can then be expanded in an infinity of ways into the series

$$Q_0(1,\tau) = \sum_{n=0}^{\infty} (p_n' \cos n\tau + p_n'' \sin n\tau) \tag{7.4}$$

which is valid in $0 \le \tau \le \pi$.*

Conditions 1) and 2), combined with (7.3′) and (7.3″), are sufficient to determine $h_n(\rho, \tau)$ uniquely.

$$h_{2n}(\rho,\tau) = \rho^n \cos n\tau \qquad (n = 0, 1, 2, \ldots) \tag{7.6}$$

is obtained quite straightforwardly. The functions h_{2n-1} are somewhat more cumbersome to compute. Taking into account property 2),

$$h_{2n-1}(1,\tau) = \begin{cases} \sin n\tau & \text{if } 0 \le \tau < \pi, \\ -\sin n\tau & \text{if } \pi \le \tau < 2\pi. \end{cases}$$

By Fourier series expansion of $h_{2n-1}(1, \tau)$, and matching with harmonic functions term-by-term,

$$h_{2n-1}(\rho,\tau) = \frac{4n}{\pi} \sum_{k=0}^{\infty} \epsilon_k \frac{\lambda_{n-k}}{n^2 - k^2} \rho^k \cos k\tau \tag{7.7}$$

$(n = 1, 2, \ldots)$, where $\epsilon_0 = \frac{1}{2}$, $\epsilon_1 = \epsilon_2 = \ldots = 1$; $\lambda_\nu = 0$ for even values of ν and $\lambda_\nu = 1$ for odd values of ν.

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*Actually, a function $\Phi(\tau)$ with continuous $\Phi'(\tau)$ in $(0, \pi)$, can be expanded into series of the forms

$$\Phi(\tau) = \sum_0^{\infty} a_n \cos n\tau \tag{7.5'}$$

and

$$\Phi(\tau) = \sum_1^{\infty} b_n \sin n\tau. \tag{7.5''}$$

The redundancy (and consequent greater flexibility) in the expansion (7.4) allows a closer approximation of $Q^*(1, \tau)$ then would be possible with series of either the type (7.5′) or of (7.5″). The latter series converge in the neighborhood of the endpoints $0, \pi$ rather slowly in general, because either all the base functions or all their first derivatives vanish at the end points. Therefore the approximation of a function with non-zero values and first derivatives of 0 and π is not practical with the series (7.5′) or (7.5″).

The corresponding complex forms are

$$H_{2n}(s) = s^{n} \qquad (n = 0, 1, 2, \ldots) \tag{7.8}$$

$$H_{2n-1}(s) = \frac{4n}{\pi} \sum_{k=0}^{\infty} \epsilon_k \frac{\lambda_{n-k}}{n^2 \cdot k^2} s^{k} \qquad (n = 1, 2, \ldots) \tag{7.9}$$

The expansions (7. 7), (7. 9) converge too slowly if ρ is close to unity. Therefore, it was necessary to use the following alternative expression:

$$H_{2n-1}(s) = \frac{2}{\pi}\left\{\frac{1}{2}(s^{n} - s^{-n}) \ln \frac{1-s}{1+s} + \sum_{k=1}^{n-1} \frac{\lambda_k}{k} (s^{n-k} - s^{-(n-k)})\right\} \tag{7.10}$$

Equation (7. 10) was obtained from (7. 9) by a tedious rearrangement of the series. The equivalence of (7. 9) and (7. 10) is simpler to show by substituting the power-series expansion of $\ln \frac{1+s}{1-s}$ into (7. 10). The two series are complementary i. e. , (7. 10) is suitable for computation when $|s|$ is not too small, and (7. 9) should be used for small $|s|$.

<u>Variational integrals.</u> Approximating Q_0 with a finite number of terms of the series (7. 2), both $\tilde{\Delta}[\tilde{Q}]$ and $\tilde{B}[\tilde{Q}]$ become functions of the coefficients p_n. We write

$$\left.\begin{aligned} \tilde{\Delta}[\tilde{Q}] &\cong \delta(p_0, \ldots, p_{N-1}), \\ \tilde{B}[\tilde{Q}] &\cong \beta(p_0, \ldots, p_{N-1}) \end{aligned}\right\} \tag{7.11}$$

and

$$\tilde{\tilde{F}}[\tilde{\tilde{Q}}] \cong \Phi(p_0, \ldots, p_{N-1}) \equiv \delta(p) + \beta(p).$$

The functions δ, β can be obtained from (6. 3″) and (6. 5) respectively, by the substitution of

$$Q_0(\rho, \tau) \cong \sum_{n=0}^{N-1} p_n h_n(\rho, \tau), \tag{7.12}$$

Thus, $\delta(p)$ is a mixed quadratic form,

$$\delta(p) = \frac{1}{2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} a_{mn} p_m p_n + \sum_{n=0}^{N-1} b_n p_n , \tag{7.13}$$

The coefficients a_{mn}, b_n are defined by the formulas

$$a_{mn} = \int_0^{\pi} h_m(1,\tau) K_n(\tau) d\tau , \tag{7.14}$$

$$b_n = \frac{1}{2} \int_0^{\pi} h_n(1,\tau) d\tau , \tag{7.15}$$

and $K_n(\tau) = \left[\frac{\partial}{\partial \rho} h_n(\rho,\tau)\right]_{\rho=1}$. From Eqs. (7.8) and (7.10)

$$K_{2n}(\tau) = n \cos n\tau , \tag{7.16'}$$

$$K_{2n-1}(\tau) = \frac{2}{\pi} \left\{ n \cos n\tau \ln \frac{\tau}{2} + \frac{\sin n\tau}{\sin \tau} + 2 \sum_{k=1}^{n-1} \lambda_{n-k} \frac{k}{n-k} \cos k\tau \right\} , \tag{7.16''}$$

The matrix elements a_{mn}, b_n can be expressed from (7.14) and (7.15) in closed form. Since this involves a considerable amount of transformation of series, and is quite laborious, the details are omitted. The closed formulas obtained for the matrix elements are very useful in the numerical work, and are listed in Appendix 4. These formulas are consistent with the symmetry of the matrix a_{mn}, and were checked by numerical integration.

No similar reduction of $\beta(p)$ is possible because of the non-linearity of the boundary conditions. Thus, from (6.5)

$$\beta(p) = \int_0^{\pi/2} E(t,p)^{-1} [\cos t . E(t,p) - 1]^2 \sin^{-3} t \, dt \tag{7.17}$$

where

$$E(t,p) = \exp\left(\sum_0^{N-1} p_n h_n(1,2t)\right) . \tag{7.18}$$

The problem of finding the function $\tilde{Q}(\rho,\tau)$, for which $F[\tilde{Q}]$ assumes its minimum is thus reduced (in approximation) to that of finding the minimum of the function of N variables $\Phi(p) = \delta(p) + \beta(p)$.

The function $\Phi(p)$ will be minimized by constructing a sequence of "admissible" coefficient families $\{p_n^{(r)}\}$ $(r = 1, 2, 3, \dots)$ such that

$$\Phi(p^{(r)}) \to \inf \Phi(p) \quad \text{as} \quad r \to \infty .$$

In the numerical work for the determination of the function Q, the coefficients $p_n^{(r)}$ were subjected to the constraints given below. A combination of coefficients will be considered admissible only if the function

$$G_0^{(r)}(\rho, \tau) = \sum_n p_n^{(r)} h_n(\rho, \tau)$$

satisfies the boundary conditions at $s = \pm 1$; i.e. the conditions (b′), (c′), and (f′) in § 2. Condition (f′) (Eq. (6.10)) implies

$$\sum_k p_{2k}^{(r)} = C, \tag{7.19}$$

$$\sum_k k\, p_{2k-1}^{(r)} = 0, \quad {}^{*} \tag{7.20}$$

$$\sum_k k^2 p_{2k}^{(r)} = -\frac{1}{4} . \quad {}^{**} \tag{7.21}$$

These equations already imply the milder conditions (c′), considering the forms (7.7), (7.9) of $H_n(s)$.

Condition (b′): From the form (3.9) of $H_{2n-1}(s)$ it follows that

$$H_{2n-1}(s) = C_n + \frac{2}{\pi}(-1)^n n(s+1)\ln(s+1) + O(|s+1|)$$

as $s \to -1$; hence,

$$\Gamma_0(s) = C + \frac{2}{\pi}\sum_n (-1)^n n\, p_{2n-1}(s+1)\ln(s+1) + O(|s+1|) .$$

---

*Note that constraint (7.20) guarantees that no term of the form $A(s-1)\ln(s-1)$ will appear in the expansion for $s \to +1$.

**I am indebted to R. C. Ackerberg for the formulation (7.21).

Thus the condition (b′) is already satisfied. Nevertheless the asymptotic equation (4.2) of Part I implies that $(d/ds)\,\Gamma_o(s)$ is bounded in the neighborhood of the point $s = -1$. Hence in the numerical computation we may (and did) impose the additional constraint

$$\sum_k (-1)^k \, k \, p^r_{2k-1} = 0 . \tag{7.22}$$

8. NUMERICAL METHOD

We will denote

$$B_n(p) \equiv \frac{\partial\phi}{\partial p_n} = b_n + \int_0^{\pi/2} \{\cos^2 t \; E(t,p) - E(t,p)^{-1}\} h_n(1,2t) \frac{dt}{\sin^3 t}, \tag{8.1}$$

and

$$A_{mn}(p) \equiv \frac{\partial^2\phi}{\partial p_m \partial p_n} =$$

$$= a_{mn} + \int_0^{\pi/2} \{\cos^2 t \; E(t,p) + E(t,p)^{-1}\} h_m(1,2t) h_n(1,2t) \frac{dt}{\sin^3 t} \tag{8.2}$$

These integrals converge if (7. 19), (7. 22) are satisfied.

With the constraints (7. 19) - (7. 22) added, the variational problem implies by the use of the Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$B_n(p) + m_{n1}\lambda_1 + m_{n2}\lambda_2 + m_{n3}\lambda_3 + m_{n4}\lambda_4 = 0. \tag{8.3}$$

The coefficients $m_{n\nu}$ are listed in Table 1.

n	m_{n1}	m_{n2}	m_{n3}	m_{n4}
=2k	1	0	k^2	0
=2k-1	0	k	0	$(-1)^k k$

Table 1

In vector notation (8. 3) can be written

$$\bar{B}(\bar{p}) + \bar{\bar{M}}\,\bar{\lambda} = 0 \tag{8.4}$$

where $\bar{\bar{M}}$ is a (known) N x 4 matrix, $\bar{\lambda}$ a 4-element column matrix.

The solution of (8. 2) was obtained by Newton's method. In principle, this required choosing a starting vector $p^{(o)}$ which satisfies the constraints (7. 19) through (7. 22), and to compute the sequences $p^{(r)}, \lambda^{(r)}$, $(r = 1, 2, 3, \ldots)$ defined recursively by the following relations:

$$\bar{\bar{A}} (\bar{p}^{(r+1)} - \bar{p}^{(r)}) + \bar{\bar{M}} \bar{\lambda}^{(r+1)} = - \bar{B} (\bar{p}^{(r)}) , \tag{8.5}$$

$$\bar{\bar{M}}^{*} (\bar{p}^{(r+1)} - \bar{p}^{(r)}) = 0 , \tag{8.6}$$

where $\bar{\bar{M}}^{*}$ is the transpose of $\bar{\bar{M}}$. Equation (8. 6) ensures that the constraints (8. 3) continue to be satisfied. Equations (8. 5) and (8. 6) represent a linear system of $N+4$ equations, with the unknowns $p_n^{(r+1)} - p_n^{(r)}$, $(n = 0, \ldots, N-1)$ and $\lambda_j^{(r+1)}$, $(j = 1, 2, 3, 4)$. It can be shown that the determinant of the system cannot vanish so that (8. 5) - (8. 6) always has a unique solution.

The iteration for the computation of the coefficients p_n was programmed for the IBM 7040 computer using this method. The computation was started with $N = 4$, the lowest number for which the constraints can be satisfied. For $N = 4$, the coefficients p_n are fully determined by the constraints. Then a number of iterations were carried out for $N = 5$, with the starting value $p_4 = 0$; when the function $\Phi(p_o, \ldots, p_4)$ reached its minimum for $N = 5$ with sufficient approximation, the iteration was continued with $N = 6$, with a starting value $p_5 = 0$, and so on. Thus, a double iteration was used, consisting of a sequence of iteration steps ("major steps"), and each major step was itself an iteration for a fixed value of N_o.

Practical details of the computation.

(1) In each step of the iteration $N(N+1)/2$ elements of the matrix A and N elements of b have to be evaluated by numerical integration. In each integration the integrand has to be evaluated in M places, where M is the number of subdivisions of the integration interval $(0, \frac{\pi}{2})$. As a result, the numerical integration accounts for most of the computer time needed to perform the Newton iteration. We used in the integrations

Gauss mechanical quadrature, 96-point formula.[7] The integration subroutine yielded 5 or more significant digits accuracy, when the integrands were bounded, reasonably smooth functions. To test the integration routine, a_{mn}, b_n were evaluated with numerical integration, and the results were compared with the closed formulas derived. (Appendix 3)

(2) The error of each step of Newton's iteration was measured by

$$\epsilon = \max_k |B_k(p)| / \max_k |S_k|$$

where S_k is the biggest partial sum (in absolute value) in the computation of B_k. The iteration in a major step was considered finished when ϵ did not exceed 2×10^{-6}. It was somewhat surprising that with the exception of the first three major steps, this limit was reached in a single iteration. At the end of each major step, we computed the new approximation to Q_o and the error $\delta(t)$ committed in the boundary condition (5. 1), in the 96 subdivision points of integration, i. e.,

$$\delta(t) = \sinh Q(\varphi, 0) + Q_\varphi(\varphi, 0) = \sinh \tilde{Q}(1, 2t) + \tilde{Q}_\rho(1, 2t) \frac{\sin^3 t}{2\cos t}$$
$$= \frac{1}{2\cos t}\left\{\cos^2 t\, E(t,p) - E(t,p)^{-1} + \left[1 + 2\sum_0^{N-1} p_n K_n(1, 2t)\right]\sin^3 t\right\}.$$

The square norm of $\delta(t)$.

$$\|\delta\| = \left\{ 2\int_0^{\pi/2} \delta(t)^2 \frac{\cos t}{\sin^3 t}\, dt \right\}^{1/2}$$

was used to judge the goodness of the answer in each major step. As an additional check the total angle of deflection of the jet was computed after each major step using the formula

$$\theta_{max} = \int_0^\infty \sinh Q(\varphi, 0)\, d\varphi$$
$$= -\int_0^{\pi/2} \left\{\cos^2 t\, E(t,p) - E(t,p)^{-1}\right\} \frac{dt}{\sin^3 t} ,$$

(The theoretical limit value of θ_{max} after infinitely many iterations is of course 90°). The values of $\Phi(p)$, $||\delta||$, θ_{max} obtained in the course of the iteration are listed in Table 2. As is evident from this table, $||\delta||$ decreased steadily during the first 11 major steps, and reached a minimum of 4.38×10^{-4}, well within the tolerance of engineering computations. After this the goodness of the approximation, as judged from the values of $||\delta||$, deteriorated first slowly, then faster, and in general changed in an erratic fashion. The reason for this peculiar trend is probably that the integration routine loses accuracy for high harmonics $\sin n\tau$, $\cos n\tau$, while the theoretical gain in accuracy of these terms is becoming smaller. Also, the SHARE routine used for the solution of systems of linear equations loses accuracy for a large number of equations. Another measure of the progress of the computation is the decrease of $\Phi(p)$. As seen from Table 2, $\Phi(p)$ converges extremely rapidly to its minimum, in fact much more rapidly than δ. (The best value of $\Phi(p)$ was 0.474498, whereas the first (!) major step resulted in $\Phi(p) \sim 0.480\ldots$) The coefficients $A_{ij}(p)$ also converge rapidly; they hardly change from one step to the next, once computed. However, the coefficients p_i do not show any trend of convergence, a fact explained by the infinitely many possibilities to represent a single function $Q_o(1, \tau)$ by a series $\sum p_n h_n(1, \tau)$. The "best" choice of the coefficients p_n $(n = 0, 1, \ldots, 14)$ is listed in Table 3.

The iteration described above took 5½ minutes on the IBM 7040 computer.

(3) In the final phase of the computation the coefficients p_n were used to obtain the functions $X(\varphi, \psi)$ and $Y(\varphi, \psi)$. This was based on the formula

$$Z = \int_0^w \left[e^{-\Gamma_o(S(w))} - C \right] \frac{2\,dw}{S(w)+1} + C\left[w - 2(-w)^{1/2} \right] \qquad (8.7)$$

where

$$C = e^{-\Gamma_o(-1)} .$$

Equation (8.7) is a direct consequence of Part I, Eq. (4.5). This form of the equation has the advantage that the integrand remains bounded at the origin. The integration was carried out by a subroutine to solve systems of ordinary differential equations, based on the Adams-Bashford method (4th order), written by Kenneth Plotkin.

N	ν	$\phi(p)$	$\|\delta\|$	θ_{max}/degr.	T_1/sec.
5	5	0.48025144	3.807×10^{-1}	66.58	.68
6	2	0.47479781	1.671×10^{-1}	93.664	.90
7	2	0.47450213	2.347×10^{-2}	87.932	1.17
8	1	0.47449958	3.823×10^{-3}	88.223	1.48
9	1	0.47449929	7.608×10^{-3}	88.6648	1.83
10	1	0.47449854	1.631×10^{-3}	89.5812	2.23
11	1	0.47449850	4.156×10^{-3}	89.6775	2.68
12	1	0.47449845	1.017×10^{-3}	89.7799	3.18
13	1	0.47449843	4.63×10^{-4}	89.992	3.73
14	1	0.47449842	4.62×10^{-4}	89.968	4.32
15	1	0.47449841	4.38×10^{-4}	89.974	4.97
16	1	0.47449841	1.676×10^{-3}	90.027	5.65
17	1	0.47449841	1.587×10^{-3}	89.979	6.37
18	1	0.47449839	5.88×10^{-4}	89.921	7.17
19	1	0.47449838	1.640×10^{-3}	89.889	8.00
20	1	0.47449838	2.257×10^{-3}	89.899	8.88

Table 2.

N	number of unknown coefficients p_n
ν	number of iterations/major step
$\phi(p)$	value of variational integral
$\|\delta\|$	norm of error in boundary condition
θ_{max}	total angle of deflection of jet
T_1	time required for individual iteration

n	p'_n	p''_n
0	$.520853172 \times 10^{-1}$	
1	$-.433968985 \times 10^{-2}$	$-.106932819 \times 10^{-2}$
2	$-.261440941 \times 10^{-1}$	$-.158038496 \times 10^{-1}$
3	$-.225389441 \times 10^{-1}$	$-.210494106 \times 10^{-2}$
4	$-.410373311 \times 10^{-2}$	$.102374660 \times 10^{-1}$
5	$.461629598 \times 10^{-2}$	$.168663154 \times 10^{-2}$
6	$.676816046 \times 10^{-3}$	$-.155702682 \times 10^{-2}$
7	$-.251967831 \times 10^{-3}$	$-.149858572 \times 10^{-3}$

Table 3.

$(p_{2n} = p'_n \,,\; p_{2n-1} = p''_n)$

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PART III

by

Robert C. Ackerberg and Alexander Pal

9. DISCUSSION OF RESULTS AND CONCLUSIONS

Plots of the bounding streamline which were determined numerically using a variational technique are displayed in Figs. 3 and 5, along with Taylor's theoretical results [1] and some experimental data from Vizel and Mostinskii [3]. Some of the streamlines and equipotential lines are shown in Fig. 6 in the neighborhood of the jet opening. The numerical results indicate a deeper jet penetration than had been found by either Taylor or experiment. Taylor mentions the difficulty of verifying theoretical results experimentally due to the viscous spreading of the jet which would fill a wedge of nearly 40 degrees. With this in mind, the discrepancy between the theoretical curve and the experimental data in Fig. 5 is probably not so great.

Finally, using Bernoulli's principle in the external flow, the coefficient of the pressure can be written

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_2 q_{\infty 2}^2} = 1 - e^{2\Omega(\varphi,\psi)} . \tag{9.1}$$

Numerical values for C_p were computed versus distance along the plate upstream and are displayed in Fig. 7. It is readily seen that the pressure decreases slowly from the stagnation pressure at the jet exit to the free stream value many slot widths upstream.

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APPENDIX 1

Proof of Equation (5. 4):

$$4\Delta_-[h,k] = P\int_{-\infty}^{+\infty} h_\psi(\varphi,0)\,k(\varphi,0)\,d\varphi .$$

To show (5. 4) we transform $\Delta_\epsilon^\lambda[h,k]$ by Green's formula, taking into account that $h(\varphi,\psi)$ is harmonic. Thus,

$$D_\epsilon^\lambda[h,k] \equiv \frac{1}{2}\iint_{G_\epsilon^\lambda} \nabla h\,\nabla k\,d\varphi\,d\psi = \frac{1}{2}\oint_{\partial G_\epsilon^\lambda} \frac{\partial h}{\partial n}\,k\,ds ,$$

and

$$\left.\begin{aligned} \Delta_\epsilon^\lambda[h,k] &= D_\epsilon^\lambda[h,k] + \frac{1}{2\pi\epsilon^2\ln\epsilon}\int_{C_\epsilon} h\,ds\int_{C_\epsilon} k\,ds \\ &= \frac{1}{2}\left\{\int_{-\lambda}^{-\epsilon} + \int_{\epsilon}^{\lambda}\right\} h_\psi(\varphi,0)\,k(\varphi,0)\,d\varphi \\ &\qquad + \frac{1}{2}\int_{C_\lambda} h_r\,k\,ds - \frac{1}{2\epsilon}\int_{C_\epsilon}\chi\,k\,ds , \end{aligned}\right\} \quad \text{(A1. 1)}$$

where

$$\chi = \epsilon\,h_r(\varphi,\psi) - \frac{1}{\pi\epsilon\ln\epsilon}\int_{C_\epsilon} h\,ds .$$

Let now $\epsilon \to 0$. Then, using property (b), a simple computation shows that $\chi \to 0$, and since k is bounded,

$$\frac{1}{2\epsilon}\int_{C_\epsilon}\chi\,k\,ds \to 0 .$$

Letting $\lambda \to \infty$, from property (c) follows that $\int_{C_\lambda} h_r\,k\,ds \to 0$. Substituting these limits into (A1. 1), the statement (5. 4) follows.

APPENDIX 2

Proof of the identity (5.6):

$$\Delta[u] = D_\lambda[u] + D^\lambda[u^*] + \frac{\alpha}{2\pi\lambda}\int_{C_\lambda} u\,ds - \frac{\alpha^2}{2\pi}\ln\lambda \; .$$

$\Delta_\epsilon^\lambda[u]$ can be written by the substitution $u = u^* + \frac{\alpha}{\pi}\ln r$, using polar coordinates (r, θ), in the form

$$4\,\Delta_\epsilon^\lambda[u] = \iint_{G_\lambda^\epsilon}\left[\left(\frac{\partial u^*}{\partial r} + \frac{\alpha}{\pi r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u^*}{\partial\theta}\right)^2\right] r\,dr\,d\theta$$

$$+ \frac{1}{2\pi\,\ln\epsilon}\left\{\int_{-\pi}^{\pi}\left[u^*(\epsilon,\theta) + \frac{\alpha}{\pi}\ln\epsilon\right]d\theta\right\}^2 =$$

$$= \iint_{G_\epsilon^\lambda}(\nabla u^*)^2\, r\,dr\,d\theta + \frac{2\alpha}{\pi}\int_{-\pi}^{\pi} u^*(\lambda,\theta)\,d\theta$$

$$+ \frac{1}{2\pi\,\ln\epsilon}\left\{\int_{-\pi}^{\pi} u^*(\epsilon,\theta)\,d\theta\right\}^2 + \frac{2\alpha^2}{\pi}\ln\lambda \; .$$

Let now $\epsilon \to 0$. Taking into account that u^* is bounded in the neighborhood of the origin, we obtain

$$4\Delta^\lambda[u] = 4D^\lambda[u^*] + \frac{2\alpha}{\pi}\int_{-\pi}^{\pi} u^*(\lambda,\theta)\,d\theta + \frac{2\alpha^2}{\pi}\ln\lambda \; .$$

The proposition follows now from the identity

$$\Delta[u] = \Delta^\lambda[u] + D_\lambda[u] \; .$$

APPENDIX 3

Proof of the convexity of the functional $F[u]$.

We first show that the (ordinary) Dirichlet-integral is a convex functional. Let S denote an arbitrary domain, $D_s[u]$ the Dirichlet-integral of the function u in S and $d[u] = (D_s[u])^{\frac{1}{2}}$, the "Dirichlet-norm" of u. If $d[u] < \infty$, $d[v] < \infty$, then u, v satisfy the following the triangle inequality[8]

$$d[u+v] \leq d[u] + d[v] . \tag{A3.1}$$

Let k by any number, $0 < k < 1$. Then, since $d[u]$ is homogeneous first order in u,

$$\Phi(k) \equiv d[ku + (1-k)v] \leq k\, d[u] + (1-k)\, d[v] , \tag{A3.2}$$

i.e., $d[u]$ is convex (but not strictly, since equality is possible in (A3.2)). Moreover, since u, v are arbitrary, $\Phi(k)$ is a convex, non-negative function. Actually, $\Phi(k)$ can vanish at most for a single value of k, for which $ku + (1-k)v = 0$ almost everywhere (possible only if u and v are linearly dependent). This implies that $[\Phi(k)]^2$ is strictly convex; consequently, if $u \neq v$,

$$[\Phi(k)]^2 < k[\Phi(0)]^2 + (1-k)[\Phi(1)]^2 , \tag{A3.3}$$

i.e.,

$$D_S[ku + (1-k)v] < k\, D_S[u] + (1-k)\, D_S[v] .$$

Apply now (A3.3) to $S = G_\lambda$, and the admissible functions u, v, $u \neq v$, and also to $S = G^\lambda$, u^*, v^*. Then from (5.6) follows the convexity of $\Delta[u]$.

The convexity of

$$B[u] = \int_0^\infty [\cosh u(\varphi, 0) - 1]\, d\varphi$$

is an immediate consequence of the convexity of the function $\cosh u$. Therefore $F[u] = \Delta[u] + B[u]$ is also convex.

Evaluation of the Matrices (a_{mn}), (b_n).

$$a_{2m,2n} = 0 \quad \text{if } m \neq n ;$$

$$a_{2n,2n} = \frac{\pi}{2} n ;$$

$$a_{2m,2n-1} = \lambda_{m-n} \frac{2mn}{n^2 - m^2} ,$$

($\lambda_k = 0$ if k is even, $= 1$, if k is odd)

$$a_{2m-1,2n-1} = -\frac{8}{\pi} \frac{mn}{m^2-n^2} (\omega_m - \omega_n) \quad \text{if } m \neq n ,$$

$$a_{2n-1,2n-1} = \frac{4n}{\pi} \Omega_n .$$

Here

$$\omega_{2k} = 1 + \frac{1}{3} + \dots + \frac{1}{2k-1} ,$$

$$\omega_{2k+1} = 1 + \frac{1}{3} + \dots + \frac{1}{2k-1} + \frac{1}{2} \frac{1}{2k+1} ,$$

$$\Omega_{2k} = 1 + \frac{1}{3^2} + \dots + \frac{1}{(2k-1)^2} ,$$

$$\Omega_{2k+1} = 1 + \frac{1}{3^2} + \dots + \frac{1}{(2k-1)^2} + \frac{1}{2} \frac{1}{(2k+1)^2} .$$

The matrix $\bar{\bar{a}}$ is symmetric, i.e., $a_{mn} = a_{nm}$. For both even and for odd subscripts only non-negative values may be substituted.

The matrix $\bar{b}$:

$$b_0 = \frac{\pi}{2} , \quad b_{2k} = 0 \quad \text{if } k > 0 .$$

$$b_{4k+1} = \frac{1}{2k+1} , \quad b_{4k+3} = 0 .$$

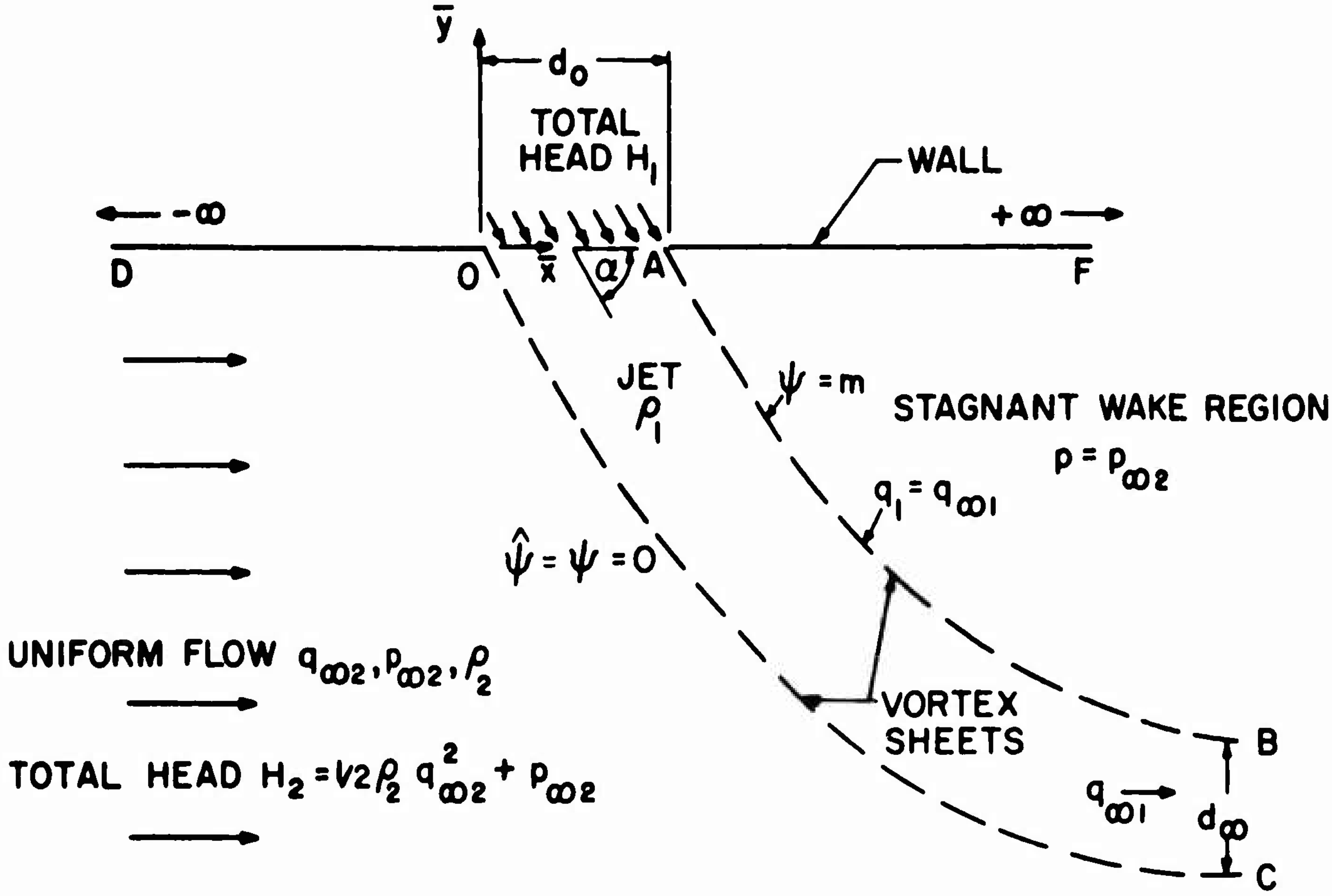


FIG. 1a. REGION OF FLOW IN PHYSICAL PLANE

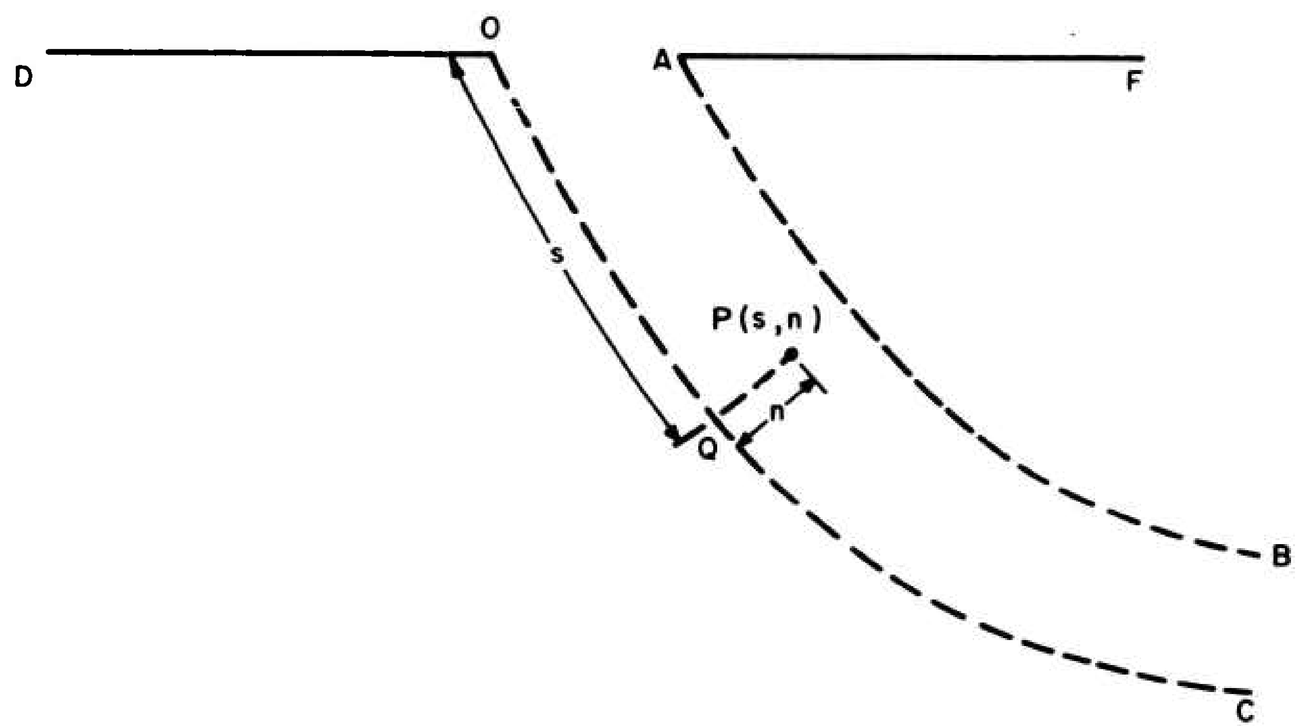


FIG. 1b. COORDINATE SYSTEM IN THE JET

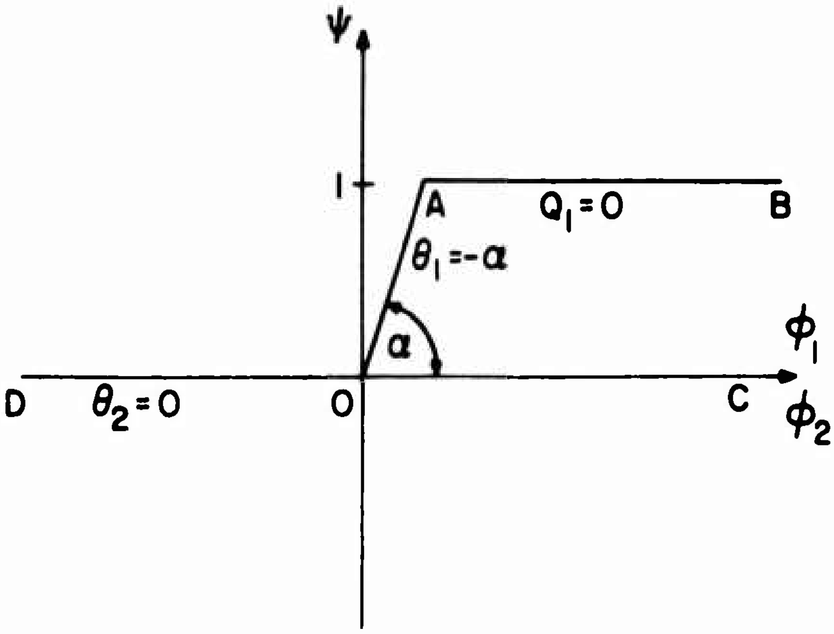


FIG. 2 w_1 - w_2 - PLANES

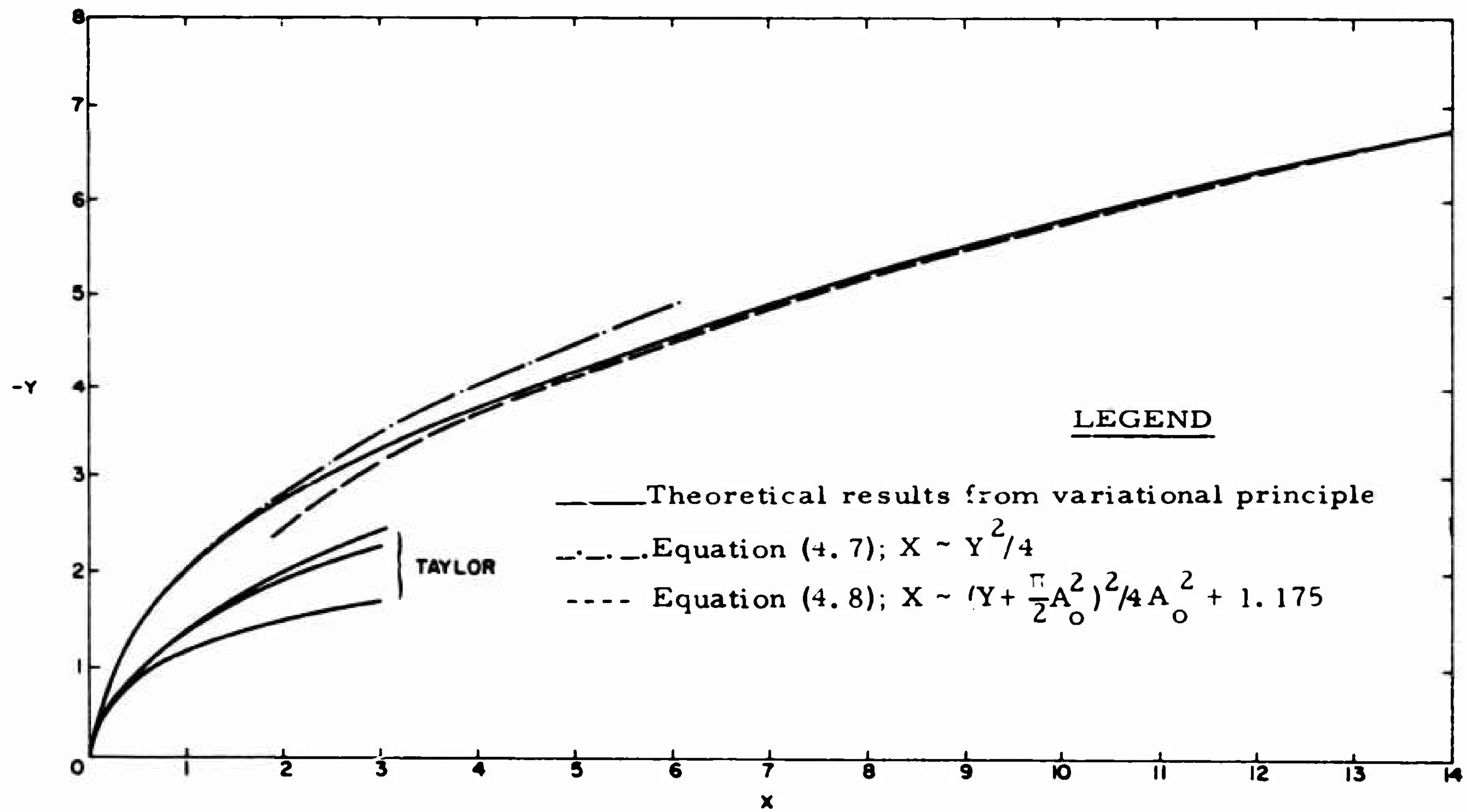
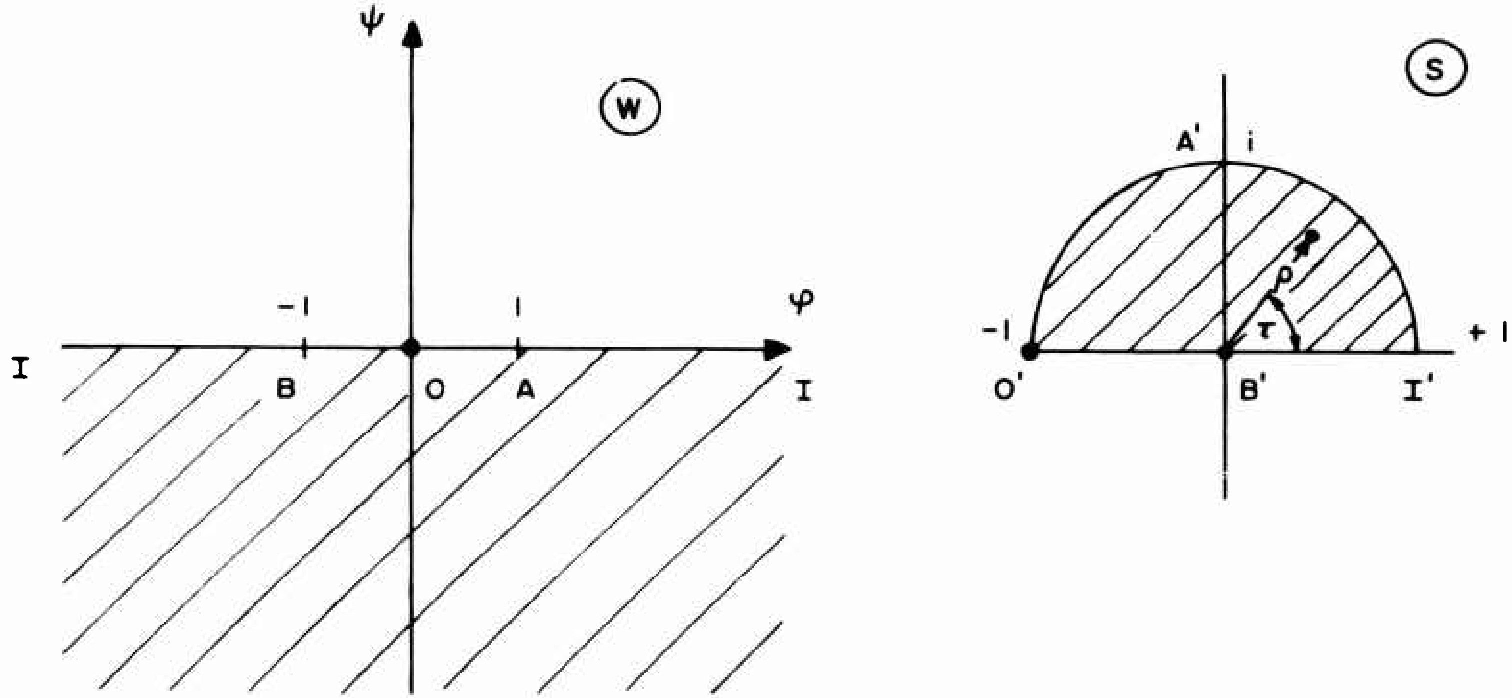


FIG. 3. THE BOUNDING STREAMLINE



G. 4. THE CONFORMAL MAPPING $w = -\left(\frac{s+1}{s-1}\right)^2$

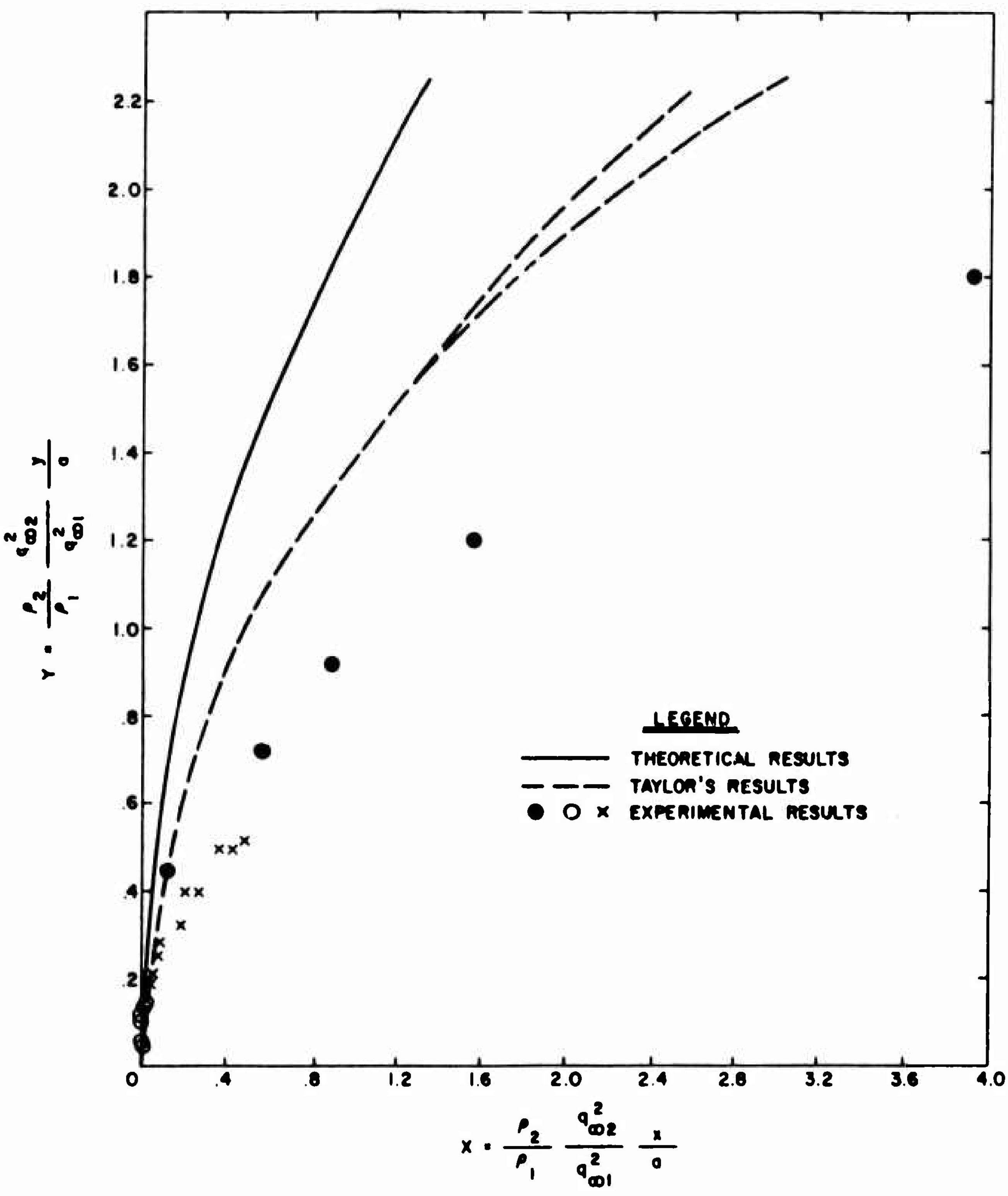


FIG. 5. THE BOUNDING STREAMLINE NEAR THE JET EXIT

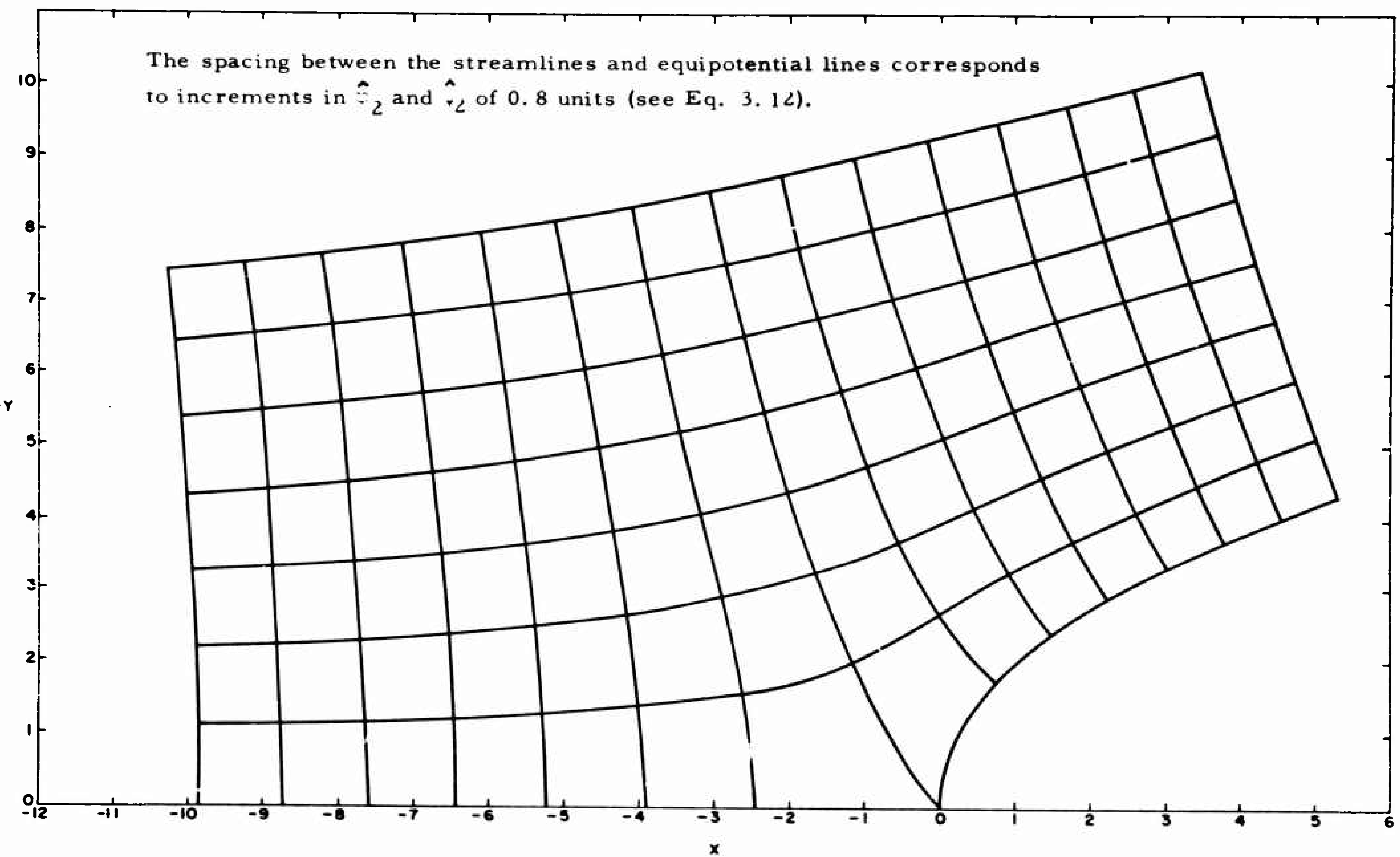


FIG. 6. STREAMLINES AND EQUIPOTENTIAL LINES

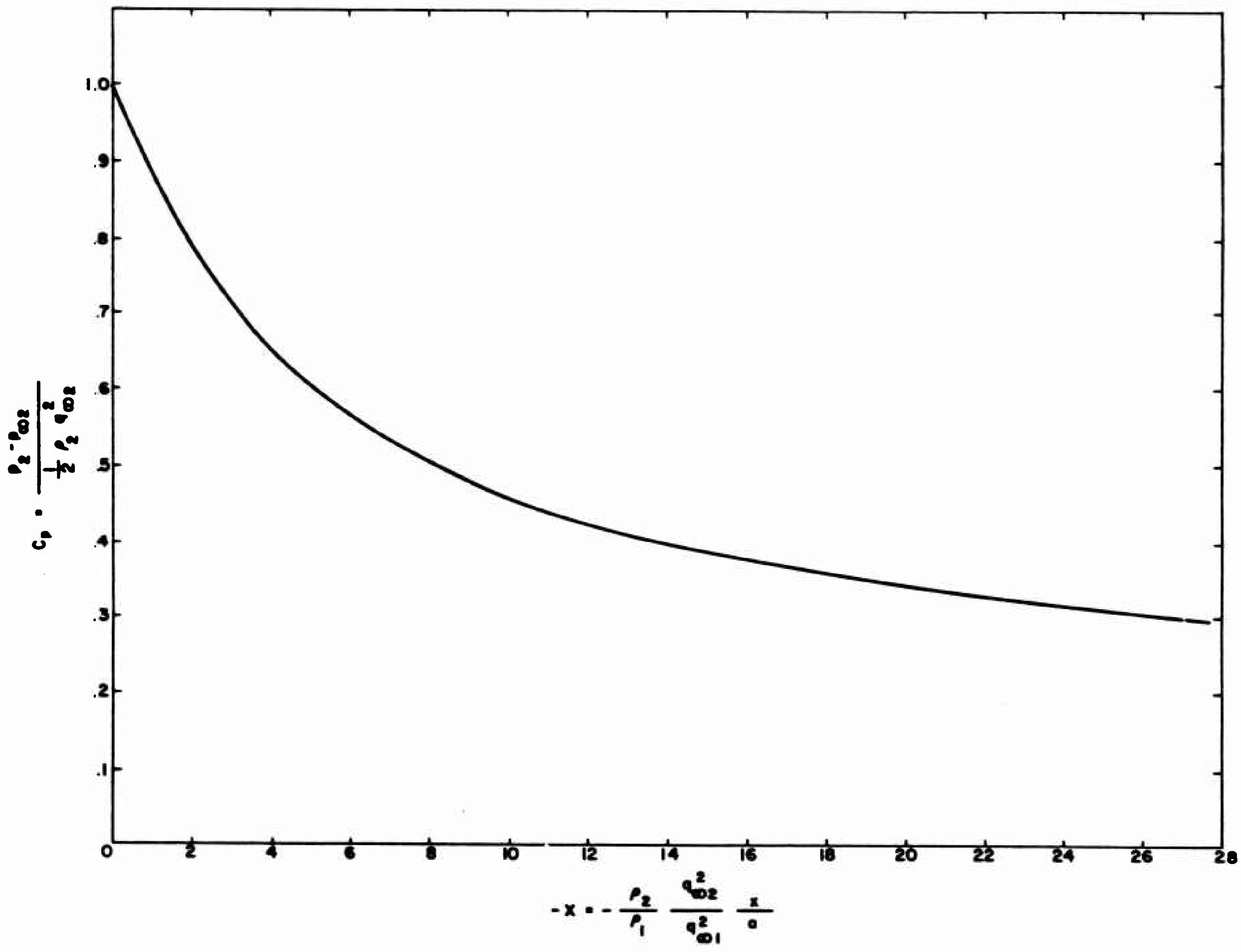


FIG. 7. COEFFICIENT OF THE PRESSURE